Understanding Rate and Unit Rate: Dragonfly

Math Goals
This lesson is intended to help your students:

➔ Extend their understanding of ratios and proportional relationships
➔ Work with situations involving constant rates of change

Common Core State Standards
This lesson involves a range of mathematical practices from the standards, with emphasis on the ability to do the following:

➔ Looking for and using structure to recognize rates, and proportional relationships (MP7)
➔ Describe and interpret descriptions of rates, and proportional relationships using precise language (MP6)

This lesson also asks students to select and apply mathematical content from across the grades, including the content standards:

➔ Recognize and represent proportional relationships between quantities. (7.RP2)
➔ Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. (7.RP2a)
➔ Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. (7.RP2b)
➔ Solve unit rate problems including those involving unit pricing and constant speed. (6.RPR.3b)

Introduction
The purpose of this lesson is to help students make sense of problems that can be solved by

➔ figuring out the unit rate, and
➔ using rate and unit rate to determine if a relationship is proportional

The tasks in this lesson give students the opportunity to work with situations that represent three types of rate problems:

➔ How long? In this situation students need to find how long did it take the dragonfly to fly a specific distance
➔ How far? Here they need to find the distance the dragonfly flew in a given time
➔ How fast? In this case students need to figure out the rate at which the dragonfly is flying

All these represent different “lenses” for looking at the same situation; all of them related to rate and unit rate.

Materials required
➔ Handouts
The way this works: one lesson in six phases

1. Launch
   This activity is about understanding unit rates in direct-proportion problems, and about the usefulness of actually calculating these rates.

2. Pose a Problem
   We present students with a fact: a dragonfly can fly 50 feet in 2 seconds. Students then develop math questions based on that fact. Having observed the class at work, the teacher arranges groups to present their problems, highlighting the three basic forms of these rate questions.

3. Workshop
   Groups work together to solve a particular “dragonfly” problem: how long does it take the dragonfly to fly 375 feet. The teacher observes the groups working on solutions and chooses groups to make presentations that will come in increasing order of sophistication.

4. Display Diverse Thinking
   Students present their strategies. We see how they culminate in the grade-level mathematics: using a unit rate.

5. Compare, Contrast, and Connect
   The teacher helps students compare the strategies and see how elements of one solution correspond to elements of another, for example, how the unit rate (25 feet per second) could appear in a table.

6. Upgrade to Grade Level
   Students now work through analogous problems having to do with a frog and a printer. Work on these problems, coming as it does after the dragonfly lesson, gives the teacher diagnostic information to use in planning the rest of the unit.
Phase 1 - Launch

The opening piece of information is: “A common green dragonfly, the fastest insect in the world, can fly a distance of 50 feet in 2 seconds.”

Students may need some orientation to the ingredients in this statement. You could begin the lesson with a class discussion of dragonflies, 50 feet, and two seconds; or you could have students begin the lesson by addressing these issues in small groups. This is a part of sense-making: just understanding what the information means.

Do students know what a dragonfly is? Ask if students have ever seen one. Show pictures. Ask students who have seen one to show with their hands how large it is.

Then we have 50 feet. Students should know from their rulers how long one foot is, but fifty? It’s never too early to have students estimate distances they can see, such as the length of the classroom or the distance from the corner of the building to the tetherball pole. But you should prepare: find a familiar distance at the school that’s about 50 feet (20–25 steps for a typical adult) so you can help them visualize the distance.

Then, two seconds. How long is that? Students may have strategies for estimating time (“one one-thousand, two one-thousand,” etc.)

Finally, help the students visualize how fast that is. Have them picture a dragonfly flying the 50-foot distance in two seconds. Ask: can you run that fast? Can a car drive that fast? (The world’s fastest sprinters run at about 35 feet per second, which is faster than a dragonfly. But our students can’t run that fast!)
Phase 2 - Pose a problem

Handout 1: Basic Dragonfly Prompt

Introduce the handout:

A common green dragonfly, the fastest insect in the world, can fly a distance of 50 feet in 2 seconds.

That’s the fact that begins this lesson.

Once you have established that students understand that statement (see page 4), you’re ready to begin the lesson in earnest: ask the students to “make up a math question that can be answered using the information in the statement. You can provide more information in your question.” Of course, students need to answer their question as well, and show their work.

Groups can use the handout to organize their work. Notice that the handout also asks them to “Use a diagram, table, chart or other method to show this situation.”
Phase 2 - Pose a problem (cont.)

During and After the Handout:

Types of Rate Problems
As students are working on this relatively open-ended task, see if you can find an example of each of the three types of rate problem among those the students are making.

These are:

1) How long does it take to fly a [given] distance? e.g., How long does it take to fly 375 feet?

2) How far can it fly in a [given] time? e.g., How far can it fly in 7 seconds?

3) How fast is it going?

Presenting All Three Types
Have a few groups present their problems. Make sure that all three types of problems are represented. If one is missing, supply it. Help students see the three categories of problem. In these time-and-distance problem, students might classify them as:

→ Find the time (how long did it take?)
→ Find the distance (how far?)
→ Find the rate (how fast?)

Some groups may add complications to their questions and combine different types. If necessary, shorten student problems for the purposes of discussion.

Questions like, “Where is it going?” or “how big is it?” are not mathematics questions or ignore the information in the statement.

Focus on the math questions that require the information in the statement. Having students formulate word problems teaches them how to read and comprehend them. This is valuable mathematical language development for EL and all students who have trouble with word problems.

Note: State tests are dominated by word problems and the majority of students have difficulty with them.
Basic Dragonfly Prompt

1. Use a diagram, table, chart or other method to show this situation.

2. Write the question that you are going to try to answer using this information, and then show your work on that question.

Your question:

_________________________________________________________________________________
_________________________________________________________________________________
Phase 3 - Workshop

Handout 2: Three more dragonfly questions

Be ready to present these three questions to your students:
1. How long does it take to fly 375 feet?
2. How far can it fly in 20 seconds?
3. How fast is it going?

Small Groups: The Dragonfly

Beginning with solo work and then moving into pairs, pose question 1 and have students solve it before posing questions 2 and 3. You can also pose a few questions generated by students earlier. Be sure to include at least one how far, one how long did it take, and one how fast.

Do not show students how to solve the problems. The purpose of this activity is to get the variety of ways of thinking students bring to the problem out for all to see. Some solo work followed by pairs will preserve the variety. As you interact with pairs, encourage the variety—help them keep from simply accepting one strategy and disregarding the other.

Also: don’t show students your way of thinking yet. You want students to provide examples along the progression of ways of thinking. You want the roots of elementary mathematics in view so students can connect the roots to grade level concepts at the top of the tree. See progression in the next section. Productive struggle is good.
Three more dragonfly questions
Phase 4 - Display Diverse Thinking

Preparing Presentations

Every pair of students prepares a presentation of their way of thinking about the questions. The job of presenting establishes the responsibility for explaining to others in a way they can understand. Select three or four groups to present to the whole class. Select presentations that span the progression from elementary to grade level. Have students present beginning with least sophisticated and progress to the most. This will generally range from skip counting to using the unit rate.

Progression of Ways of Thinking (abbreviated version)

“more about the math” below for more complete descriptions

Adding
Most 6th grade students will use additive ways of thinking to get answers to rate problems. Such methods work fine to make sense of the quantitative situation and get the answer, if the numbers are convenient so there are not too many addition calculations.

Tables
Some students will be more organized and use tables. Table is a basic tool from 4th grade through college mathematics. There use as a reasoning and explaining tool should be encouraged. Students who have undeveloped concepts of how tables are used and read need to learn it now. Many students can read a table by 6th grade, but do not know how to use them for making sense of situations.

Multiplying
Some students rush to calculating answers. Are they learning the mathematics they need to prepare for algebra? Often they are missing the chance to learn mathematics by rushing to answers. In this problem, the important mathematics to learn is how two quantities, distance and time, vary proportionally. Co-variation is a basic building block for understanding variables and functions.

Equations
The equation \( d = rt \) expresses that distance equals (speed) times (time). Speed is a rate. In this case, 25 feet/second is the speed, so \( d = 25t \). When the units of measure are made explicit,
Some students who use the equation might use \( 50/2 \) as the rate, which is equivalent. All students should learn how this equation can be obtained from \( 50/2 = 375/x \) or from \( 375/50 = x/2 \). All should also understand how the equation generates the values in the tables.

Number lines and Graphs
Some students might use number lines, double number lines or graphs to help them think about distance traveled as a function of time traveled. This indicates thinking about the quantities as variables and is a big step toward understanding functions and variables. Use students who think this way to help all students see how to think in terms of variables.

Most student responses will cluster around the these ways of thinking, which are ordered based on the sophistication of mathematics used. Using less sophisticated mathematics often requires using more sophisticated problem solving strategies.

Less sophisticated mathematics is usually excellent for making sense of a quantitative situation. Engineers and scientists do it for a living. Einstein made sense of relativity using less sophisticated mathematics before he formulated the more elegant and general theory using sophisticated mathematics. We encourage such sense making, but do not settle for it. All students have to build on the sense they make to learn grade level mathematics.
Phase 5 -
Compare, Contrast, and Connect

Building an understanding of Unit Rate

This is the place where you will nudge students to using the grade-level mathematics—the unit rate. As the presentations proceed, ask the class about the correspondences across presentations. During the second presentation, for example, ask how parts of it correspond to parts of the first presentation. Point out where they arrive at the same number by different means, ask how that happened. After the last presentation, trace the correspondences and ask how the different ways of thinking are related. The progression pages referred to above are rich in correspondences you can anticipate. For example, a group that uses skip-counting will probably touch on the same numbers that appear in a table created by a table-oriented group.

During the fourth presentation, we have the most important correspondence: ask where the unit rate is in each way of thinking. Notice how it is hidden in the skip counting because you don’t need it to get an answer. It shows up in the table in the 1, 25 row. The unit rate is the grade level mathematics students have to learn, so skip counters need to see and understand it in the tables and the equations. And the equation solvers need to see how it relates to tables and skip counting.

If there is an important way of thinking from the progression that doesn’t emerge from a particular class, you can supply it—possibly by saying some students in another class did it this way and showing them.

Questions to ask across presentations

geführt What is the relationship between pairs of numbers in the tables?

geführt Where is the unit rate in each way of thinking?

geführt What would a slower insect look like in each way of thinking? Faster bird?

geführt Which way of thinking is best for solving any problem, like with really big or really small numbers? (e.g., How long would it take to fly 10 cm? How far in an hour?)

Why Presentations?

Mathematical communication is an important part of becoming mathematically powerful. And it’s not just good for the presenters. It is each student’s job to understand each presentation. Students are learning how to understand someone else’s thinking.
Phase 6 - Upgrade to Grade-level

Handouts 3 and 4: Frog problem and Printer problem

Small Groups: More Problems

The handouts describe two more problems that are the same form as the dragonfly problem. How you present these can vary dramatically, ranging from being a homework assignment for individuals to being another groupwork task.

Ask students to use their ways of thinking to solve the Frog and Printer problems. Have them solve each using two different ways of thinking. Select a few presentations of these problems to pull student thinking along the progression to the target mathematics.

Adjust plans for teaching the rest of the rate chapter or unit.

Now you know a lot about how your class thinks about rate problems and where they need to spend more time. Plan on spending more time on lessons where they need attention and less time on lessons where they are developing nicely already. Keep one eye on the target mathematics and another on your students’ thinking. Bringing their thinking to the target mathematics along the progression will draw their prior knowledge of mathematics into grade level concepts and prepare them for learning how rates extend into linear functions and equations in 8th grade.

However you choose to use these, they give you a chance to assess how well students have internalized the lesson. Will they compute a unit rate and use it to solve the problems?
The Frog Problem

A frog can hop at a maximum speed of about 60 feet every 4 seconds.

How far can the frog go in 30 seconds? Show your work on this problem using two or three of the methods that were discussed in the dragonfly problem.
The Printer Problem

A printer can print 3 high-quality photographs in 2 minutes.

How long will it take that printer to print 14 photos? Show 2 or 3 methods that you can use to understand and solve this problem.
This lesson is all about rates and direct proportion. We have two quantities that are related by multiplication: one is some constant number times the other.

That constant number is the rate. So we have a relationship:

\[(\text{some number}) \times (\text{rate}) = (\text{some other number})\].

This kind of relationship occurs in many settings, for example:

\[(\text{time}) \times (\text{speed}) = (\text{distance})\]

\[(\text{number of beans}) \times (\text{weight of one bean}) = (\text{weight of all the beans})\]

\[(\text{number of apples}) \times (\text{cost of one apple}) = (\text{cost for the set of apples})\]

\[(\text{number of months}) \times (\text{monthly income}) = (\text{total income})\].

In the Dragonfly activity, we focus only on the first one, often described as distance equals rate times time, or \(d = rt\). But the same mathematics applies to any similar situation.

**What's a Unit Rate as Opposed to a Rate?**

We often use the word “unit” to mean something related to the number one. A “unit circle” has a radius of one. A “unit fraction” has a 1 in the numerator.

And a unit rate is a rate with a “1” in the denominator.

If you have a dragonfly that goes 50 feet in 2 seconds you can express its speed by dividing the distance by the time:

\[\text{speed} = \frac{50 \text{ feet}}{2 \text{ seconds}}\]

This makes perfect sense: it goes 50 feet in 2 seconds, and you can use that speed to perform any calculation. But we traditionally don’t leave it that way. Traditionally, we perform the division to figure out how far the dragonfly goes in one second—that is, we get a “1” in the denominator:

\[\text{speed} = \frac{50 \text{ feet}}{2 \text{ seconds}} = \frac{25 \text{ feet}}{1 \text{ second}}\]

Then, since 25/1 is just 25, we typically omit the one—but keep the “second”—and move the units into a fraction of their own. We then further change the expression by writing the fraction with feet and seconds using the word “per.” And then we often abbreviate it:

\[\frac{25 \text{ feet}}{1 \text{ second}} = \frac{25}{1} \times \frac{\text{feet}}{\text{second}} = 25 \frac{\text{feet}}{\text{second}} = 25 \text{ feet per second} = 25 \text{ fps}\].

Any of these are a “unit rate,” and students can get confused if they don’t understand that these expressions are all the same.

To make matters worse, we call these unit rates (having a 1 in the denominator), and at the same time worry about keeping the units (the feet and seconds) straight. Imagine you’re an English Learner. Just try to distinguish “unit rate” and “units straight.” They’re related, but different.
Progression of Ways of Thinking (from page 10)

Most student responses will cluster around the following ways of thinking, which are ordered based on the sophistication of mathematics used. Using less sophisticated mathematics often requires using more sophisticated problem solving strategies.

Less sophisticated mathematics is usually excellent for making sense of a quantitative situation. Engineers and scientists do it for a living. Einstein made sense of relativity using less sophisticated mathematics before he formulated the more elegant and general theory using sophisticated mathematics. We encourage such sense making, but do not settle for it. All students have to build on the sense they make to learn grade level mathematics.

➔ Adding
Most 6th grade students will use additive ways of thinking to get answers to rate problems. Such methods work fine to make sense of the quantitative situation and get the answer, if the numbers are convenient so there are not too many addition calculations.

A common example for the dragonfly problem uses skip counting (1st and 2nd grade). So students reason, 50 in 2 seconds, 100 in 4 seconds, 150 in 6 seconds, and so on up to 400 in 16 seconds. From there, they use various tactics to get back to 375 seconds.

Some will find their way from (16, 400) by noticing that 375 is 25 less than 400 and 25 is also half of 50. They reason that if it takes 2 sec to fly 50, it will fly 25 in 1 sec. This happens to be the unit rate and can be linked to the unit rate as it comes out further along the progression.

A version of additive reasoning employs a little easy multiplication by doubling: (2, 50); (4, 100); (8, 200); (16, 400). This is nice because it shows how multiplication can save time.

➔ Tables
Some students will be more organized and use tables. Table is a basic tool from 4th grade through college mathematics. There use as a reasoning and explaining tool should be encouraged. Students who have undeveloped concepts of how tables are used and read need to learn it now. Many students can read a table by 6th grade, but do not know how to use them for making sense of situations. This they should learn.

Most table makers will use the numbers given in the problem as increments in the table. You will see

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>... +2</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100</td>
<td>150</td>
<td>200</td>
<td>250</td>
<td>... +50</td>
</tr>
</tbody>
</table>

The columns in the table will correspond to the skip counting pairs.

Others will want increments of 1 for one of the quantities,
The increments of 1 lead to the unit rate and the correspondence can be discussed during presentations. Still others will look for efficiencies and might produce something like,

\[
\begin{array}{cccccc}
50 & 100 & 200 & 300 & 400 & \ldots +100 \\
2 & 4 & 8 & 12 & 16 & \ldots +4 \\
\end{array}
\]

What these tables have in common is their organized way of displaying how increases in one quantity, time, are paired with increases in the other quantity. When the increases are constant in one quantity, they are constant in the other quantity. Increases of 1 pair with increases of 25; increases of 2 pair with increases of 50. These tables show something else:

\[
\frac{25}{1} = \frac{50}{2} = \frac{75}{3} = \frac{100}{4} = \ldots
\]

If you read each column as a fraction, then all the columns in all these tables will show equivalent fractions. This constant fraction is the constant of proportionality. When written as \(25/1\), it is the unit rate in feet per second. When written as \(1/25\) it is still a rate: 1 second per 25 feet. (And if you find the decimal, you get a unit rate: 0.04 seconds per foot.) Both are correct, although feet per second is the rate conventional way of expressing speed.

The equivalent fractions that simplify to the unit rate in these tables are at the heart of the mathematics we want students to understand going forward. This is a multiplicative way of thinking about rates. Most table makers at 6th grade will be thinking about them additively…going across. By the end of the lesson, we want all students to also think about them multiplicatively…seeing the constant rate by reading up and down.

When you get to the last presentation, spend some time having students look for the unit rate in each of the presentations. Finding the unit rate here in the table is extremely important.

→ Multiplying

Some students rush to calculating answers. Are they learning the mathematics they need to prepare for algebra? Often they are missing the chance to learn mathematics by rushing to answers. In this problem, the important mathematics to learn is how two quantities, distance and time, vary proportionally. Co-variation is a basic building block for understanding variables and functions.

Students might divide 50 into 375. What does the answer, 7.5 mean? It can be thought of as how many 50s there are in 375. Or it could be thought of as 375 is 7.5 times bigger than 50. This way of thinking, “how many times bigger,” is an important development in understanding multiplication; it is the concept of scalar multiplication—enlarging and shrinking. It is worth drawing all students’ attention to scaling in
Progression of Ways of Thinking
(continued)

this problem. Understanding multiplication by a constant rate as scaling is a very important mathematical idea.

Students might reason that if 375 feet is 7.5 times bigger than 50, then the time should be 7.5 times bigger than 2 seconds, or 15 seconds. This is reasoning by scaling up, an important mathematical step toward rates and proportionality.

Some students may have been taught to “set up a proportion and cross multiply”. They will show something like

\[
\frac{50}{375} = \frac{2}{x} \quad \text{or} \quad \frac{50}{2} = \frac{375}{x}
\]

To cross multiply to get the answer misses the mathematics. Where is the unit rate here? When 50 feet/2 seconds is calculated to 25 feet per second, the unit rate results. Then 375 feet/x seconds = 25 feet/second states that the rate is constant. 25 feet/second = 50 feet/2 seconds = 375 feet/x seconds.

Seeing these rates as equivalent is the important mathematics to get from this way of thinking about the problem. Seeing the correspondence between these equivalent rates and the entries in the tables lead to the understanding students need.

→ Equations
The equation \(d = rt\) expresses that distance equals (speed) times (time). \(Speed\) is a rate. In this case, 25 feet/second is the speed, so \(d = 25t\). When the units of measure are made explicit,

\[
\text{feet} = \frac{\text{feet}}{\text{second}} \times \text{seconds}
\]

Some students who use the equation might use 50/2 as the rate, which is equivalent. All students should learn how this equation can be obtained from \(50/2 = 375/x\) or from \(375/50 = x/2\). All should also understand how the equation generates the values in the tables.

→ Number lines and Graphs
Some students might use number lines, double number lines or graphs to help them think about distance traveled as a function of time traveled. This indicates thinking about the quantities as variables and is a big step toward understanding functions and variables. Use students who think this way to help all students see how to think in terms of variables.

Why is the graph a straight line? Why does it go through (0, 0)? Where can you find the pairs of values in the tables on the graph? Where is “50 feet in 2 seconds” on the graph? Where is the unit rate? Where is the equivalence of 25/1 = 50/2 = 100/4 = 375/15? How does the graph correspond to the equation \(d = rt\)? This group of questions is worth a whole lesson. It develops many standards in grade 6 and builds the foundation for 7 and 8.
Why “per” is so important?

When we teachers talk about proportional situations, we often use special words such as per. Even native speakers get confused by these. Be alert, and if you need to, take a moment in class to be explicit about how you use these words.

Per

Per means, “for each” or “for every.”

In one way we use it, it means that you’re going to share whatever comes before the word per equally among the things that come after. So if we have three people and twelve cookies, we ask, how many cookies per person?

In that situation, most students will see that to get the numerical answer, you divide. You get $12 \div 3 = 4$. Four cookies per person. Notice how the computation is easy but has no context. When you use the “units” and the word “per,” the context and meaning come back.

When you see a “naked computation” in a context-rich problem, ask students what it means; in this case, “the four means four cookies per person” is an excellent answer.

For other uses of per, the sharing metaphor gets a little strained. But you still divide. You divide the thing before per by the thing after.

- Cookies per person? Divide the number of cookies by the number of people.
- Miles per gallon? Divide the number of miles by the number of gallons you used.
- Feet per second? Divide the number of feet by the number of seconds.

When you perform the division, the number you get is a unit rate.

Two more things worth noticing:

First, you can turn any of these around. Suppose you wanted people per cookie. Then you divide people (3) by cookies (12) and get 0.25 or 1/4. If the cookies were sharing the people, that’s how many people each cookie would get. It still makes sense, although we don’t usually think of this situation that way. The numerical insight is that if you turn the “per” statement around, the result you get is the reciprocal.

Second, the word per is in the word percent—and cent means 100. So what is 74 percent? Simply 74 divided by 100, or 0.74.

A look ahead to high school:

If you’re plotting a line from some situation, the slope of that line is in units of the y-axis per the x-axis. For example, if you put distance on y and time on x, the slope is the speed. If you put the weight of a bunch of beans on y and the number of beans on x, the slope is grams per bean—the weight of a single bean.